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New Schwarzschild-like solutions in $f(T)$ gravity through Noether symmetries

A. Paliathanasis,[1,2,3,*] S. Basilakos,[4,†] E. N. Saridakis,[5,6,‡] S. Capozziello,[2,3,7,§] K. Atazadeh,[8,9,¶] F. Darabi,[8,**] and M. Tsamparlis[1,††]

[1]*Faculty of Physics, Department of Astrophysics - Astronomy - Mechanics University of Athens, Panepistemiopolis, Athens 157 83, Greece*

[2]*Dipartimento di Fisica, Universita' di Napoli, "Federico II", Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy*

[3]*INFN Sez. di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy*

[4]*Academy of Athens, Research Center for Astronomy and Applied Mathematics, Soranou Efesiou 4, 11527, Athens, Greece*

[5]*Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece*

[6]*Instituto de Física, Pontificia Universidad de Católica de Valparaíso, Casilla 4950, Valparaíso, Chile*

[7]*Gran Sasso Science Institute (INFN), Viale F. Crispi, 7, I-67100, L'Aquila, Italy*

[8]*Department of Physics, Azarbaijan Shahid Madani University , Tabriz, 53714-161 Iran*

[9]*Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Maragha 55134-441, Iran*

Spherically symmetric solutions for $f(T)$ gravity models are derived by the so called Noether Symmetry Approach. First, we present a full set of Noether symmetries for some minisuperspace models. Then, we compute analytical solutions and find that spherically symmetric solutions in $f(T)$ gravity can be recast in terms of Schwarzschild-like solutions modified by a distortion function depending on a characteristic radius. The obtained solutions are more general than those obtained by the usual solution methods.

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I. INTRODUCTION

Modified gravity (see for instance [1]) and dark energy model (see for instance [2]) are known as the two basic approaches to describe the observed acceleration of the universe. The former dealt with the modification of Einstein's General Relativity itself whereas the latter suggests some modifications of the cosmic fluid in Einstein's General Relativity. Essentially, the two approaches consider modifications in the l.h.s (the former) or in r.h.s. (the latter) of the cosmological field equations with respect to the picture of the Cosmological Standard Model. Amongst the variety of modified gravity theories, $f(T)$ gravity has recently received considerable amount of interest and attention. It is based on the old formulation of "Teleparallel Equivalent of General Relativity" (TEGR) [3–5] which instead of the torsion-less Levi-Civita connection uses the curvature-less Weitzenböck one, however instead of the torsion scalar T it uses $f(T)$ extensions in the Lagrangian where f is a function of T [6–8].

Although TEGR coincides completely with General Relativity both at the background and perturbation levels, $f(T)$ gravity proves to exhibit novel structural and phenomenological features. In particular, imposing a cosmological background, one can extract various cosmological solutions, consistent with the observable behavior [6–12]. Furthermore, imposing spherical geometry one can investigate the spherical, black-hole solutions for $f(T)$ gravity [13–16]. These features mean that $f(T)$ gravity can be interesting, in principle, both at cosmological and at astrophysical levels.

A crucial point in the context of $f(T)$ gravity is about the allowed classes of $f(T)$ models. The aforementioned cosmological and spherical solutions lead to several viable models, although cosmological observations [11, 17] as well as Solar System tests [12] indicate that $f(T)$ must be close to the linear form. Thus, in [18–20] the authors followed the Noether Symmtery Approach [21] in order to constrain the allowed $f(T)$ forms that are compatible with the Lemaître-Robertson-Walker (FLRW) geometry. Such an approach which is used to fully solve dynamics and also to determine exactly the corresponding Lagrangian, allowing for the Noether currents in a given geometry, is very powerful and can be applied in every gravitational or field-theoretical scenario [20–31].

The Noether Symmetry Approach has a deep physical content, since it offers a theoretical justification for the specific Lagrangian form, instead of fixing it by hand or by observations. Additionally, in modified gravitational theories, where the Birkhoff theorem is not guaranteed, the Noether approach can lead to new solution subclasses that cannot be obtained by the vacuum field equations [32].

In the present work, we apply this technique with the aim to derive new spherically-symmetric solutions for $f(T)$ gravity. Specifically, in Section II, we briefly present $f(T)$ gravity, while in Section III we construct the corresponding generalized Lagrangian formulation. In Section IV, we analyze the main properties of the Noether Sym-

*Electronic address: anpaliat@phys.uoa.gr
†Electronic address: svasil@Academyofathens.gr
‡Electronic address: Emmanuel_Saridakis@baylor.edu
§Electronic address: capozzie@na.infn.it
¶Electronic address: atazadeh@azaruniv.ac.ir
**Electronic address: f.darabi@azaruniv.edu
††Electronic address: mtsampa@phys.uoa.gr

metry Approach for $f(T)$ gravity, general geometry, and the particular case of spherically symmetric geometry. Then, in Section V, we use these results in order to obtain all possible solutions, including novel ones that could not be obtained by the standard approach. Finally, we exhibit our conclusions in Section VI.

II. $f(T)$ GRAVITY

Let us review now the basic assumptions of $f(T)$ gravity. The notation is as follows: Greek indices μ,ν,... and capital Latin indices A,B,... run over all coordinate and tangent space-time 0, 1, 2, 3, while lower case Latin indices (from the middle of the alphabet) $i,j,...$ and lower case Latin indices (from the beginning of the alphabet) a,b,... run over spatial and tangent space coordinates 1, 2, 3, respectively.

In the theory of "teleparallel" gravity, as well as in its $f(T)$ extension, the dynamical variable is the vierbein field $\mathbf{e}_A(x^\mu)$. This forms an orthonormal basis for the tangent space at each point x^μ of the manifold, that is $\mathbf{e}_A\cdot\mathbf{e}_B=\eta_{AB}$, where $\eta_{AB}=\mathrm{diag}(-1,+1,+1,+1)$. Additionally, the vector $\mathbf{e}_A$ can be analyzed with the use of its components e_A^μ in a coordinate basis, namely $\mathbf{e}_A=e_A^\mu\partial_\mu$. Finally, in such a construction, the metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x)=\eta_{AB}\,e_\mu^A(x)\,e_\nu^B(x). \qquad (1)$$

Contrary to General Relativity, which uses the torsionless Levi-Civita connection, in the present gravitational formulation one uses the curvature-less Weitzenböck connection $\overset{\mathbf{w}}{\Gamma}{}^\lambda_{\nu\mu}\equiv e_A^\lambda\partial_\mu e_\nu^A$ [33], and defines the torsion tensor as

$$T^\lambda_{\mu\nu}=\overset{\mathbf{w}}{\Gamma}{}^\lambda_{\nu\mu}-\overset{\mathbf{w}}{\Gamma}{}^\lambda_{\mu\nu}=e_A^\lambda\,(\partial_\mu e_\nu^A-\partial_\nu e_\mu^A). \qquad (2)$$

Furthermore, the contorsion tensor is defined as

$$K^{\mu\nu}{}_\rho\equiv-\frac{1}{2}\Big(T^{\mu\nu}{}_\rho-T^{\nu\mu}{}_\rho-T_\rho{}^{\mu\nu}\Big), \qquad (3)$$

and for convenience, we also introduce the tensor

$$S_\rho{}^{\mu\nu}\equiv\frac{1}{2}\Big(K^{\mu\nu}{}_\rho+\delta^\mu_\rho\,T^{\alpha\nu}{}_\alpha-\delta^\nu_\rho\,T^{\alpha\mu}{}_\alpha\Big). \qquad (4)$$

Using these quantities one can define the teleparallel Lagrangian, which is the torsion scalar [4, 5], as[1]

$$T\equiv S_\rho{}^{\mu\nu}\,T^\rho{}_{\mu\nu}=\frac{1}{4}T^{\rho\mu\nu}T_{\rho\mu\nu}+\frac{1}{2}T^{\rho\mu\nu}T_{\nu\mu\rho}-T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu. \qquad (5)$$

[1] A discussion concerning the role of Torsion in General Relativity can be found in Basilakos et al. [20].

In summary, in the present formalism, all the information concerning the gravitational field is included in the torsion tensor $T^\lambda_{\mu\nu}$, and the torsion scalar T arises from it in a similar way as the curvature scalar arises from the curvature Riemann tensor in General Relativity.

While in the teleparallel equivalent of General Relativity (TEGR) the action is just T, the idea of $f(T)$ gravity is to generalize T to a function $f(T)$. This is similar in spirit to the generalization of the Ricci scalar R in the Einstein-Hilbert action of General Relativity, to a function $f(R)$ [1]. In particular, the action of $f(T)$ gravity is written as

$$I=\frac{1}{16\pi G}\int d^4xe\,[f(T)], \qquad (6)$$

where $e=\det(e_\mu^A)=\sqrt{-g}$, G is the Newton's constant, and we have set the light speed to 1. We remark here that in some works in the literature, T is generalized to $T+f(T)$, however in the present analysis it is proved more convenient to use the above ansatz. Therefore, TEGR (and thus General Relativity) is restored when $f(T)=T$ (plus a constant if we consider also the cosmological constant term).

Variation of the action (6) with respect to the vierbein gives the equations of motion

$$\begin{aligned}&e^{-1}\partial_\mu(ee_A^\rho S_\rho{}^{\mu\nu})f_T-e_A^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}f_T\\&+e_A^\rho S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT}+\frac{1}{4}e_A^\nu f(T)=4\pi Ge_A^\rho\overset{\mathbf{em}}{T}{}_\rho{}^\nu,\end{aligned} \qquad (7)$$

where f_T and f_{TT} denote the first and second derivatives of the function $f(T)$ with respect to T, respectively. Finally, the tensor $\overset{\mathbf{em}}{T}{}_\rho{}^\nu$ stands for the usual energy-momentum tensor of perfect fluid matter.

III. GENERALIZED LAGRANGIAN FORMULATION OF $f(T)$ GRAVITY

In this section, following the technique described in [30, 34], we provide a generalized Lagrangian formulation in order to construct a theory of $f(T)$ gravity. Specifically, the gravitational field is driven by the Lagrangian density $f(T)$ in (6), which can be generalized through the use of a Lagrange multiplier. In particular, we can write it as

$$L\left(x^k,x'^k,T\right)=2f_T\bar{\gamma}_{ij}\left(x^k\right)x'^ix'^j+M\left(x^k\right)\left(f-Tf_T\right), \qquad (8)$$

where $x'=\frac{dx}{d\tau}$, $M(x^k)$ is the Lagrange multiplier and $\bar{\gamma}_{ij}$ is a second rank tensor which is related to the frame [one can use $eT(x^k,x'^k)$] of the background spacetime. In the same lines, the Hamiltonian of the system is written as

$$H\left(x^k,x'^k,T\right)=2f_T\bar{\gamma}_{ij}\left(x^k\right)x'^ix'^j-M\left(x^k\right)\left(f-Tf_T\right)\;. \qquad (9)$$

In this case, the system is autonomous and because of that ∂_τ is a Noether symmetry with corresponding Noether integral the Hamiltonian H. Additionally, since the coupling function M is a function of x^k, it is implied that the Hamiltonian (9) vanishes [35].

In this framework, considering $\{x^k, T\}$ as the canonical variables of the configuration space, we can derive, after some algebra, the general field equations of $f(T)$ gravity. Indeed, starting from the Lagrangian (8), the Euler-Lagrange equations

$$\frac{\partial L}{\partial T} = 0, \quad \frac{d}{d\tau}\left(\frac{\partial L}{\partial x'^k}\right) - \frac{\partial L}{\partial x^k} = 0\,, \tag{10}$$

give rise to

$$f_{TT}\left(2\bar{\gamma}_{ij}x'^i x'^j - MT\right) = 0, \tag{11}$$

$$x^{i\prime\prime} + \bar{\Gamma}^i_{jk}x^{j\prime}x^{k\prime} + \frac{f_{TT}}{f_T}x^{i\prime}T' - M^{,i}\frac{(f - Tf_T)}{4f_T} = 0\,. \tag{12}$$

We mention here that, for convenience, the functions $\bar{\Gamma}^i_{jk}$ are considered: they are exactly the Christoffel symbols for the metric $\bar{\gamma}_{ij}$. Therefore, the system is determined by the two independent differential equations (11),(12), and the Hamiltonian constrain $H = 0$ where H is given by Eq.(9).

The point-like Lagrangian (8) determines completely the related dynamical system in the minisuperspace $\{x^k, T\}$, implying that one can easily recover some well known cases of cosmological interest. In brief, these are:

- The static spherically symmetric spacetime:

$$ds^2 = -a^2(\tau)\,dt^2 + \frac{1}{N^2(a(\tau), b(\tau))}d\tau^2 + b^2(\tau)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{13}$$

  arising from the diagonal vierbein [2]

$$e^A_i = \left(a(\tau), \frac{1}{N(a(\tau), b(\tau))}, b(\tau), b(\tau)\sin\theta\right)\,, \tag{14}$$

  where $a(\tau)$ and $b(\tau)$ are functions which need to be determined. Therefore, the line element of $\bar{\gamma}_{ij}$ and $M\left(x^k\right)$ are given by

$$ds^2_{\bar{\gamma}} = N\left(2b\;da\;db + a\;db^2\right), \quad M(a,b) = \frac{ab^2}{N}. \tag{15}$$

- The flat Friedmann-Lemaître-Robertson-Walker spacetime with Cartesian coordinates:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{16}$$

  arising from the vierbein

$$e^A_i = (1, a(t), a(t), a(t))\,, \tag{17}$$

  where t is the cosmic time and $a(t)$ is the scale factor of the universe. In this case we have

$$ds^2_{\bar{\gamma}} = 3a\;da^2, \quad M(a) = a^3(t). \tag{18}$$

- The Bianchi type I spacetime:

$$ds^2 = -\frac{1}{N^2(a(t), \beta(t))}dt^2 + a^2(t)\left[e^{-2\beta(t)}dx^2 + e^{\beta(t)}\left(dy^2 + dz^2\right)\right], \tag{19}$$

  arising from the vierbein

$$e^A_i = \left(\frac{1}{N(a(t), \beta(t))}, a(t)e^{-\beta(t)}, a(t)^{\frac{\beta(t)}{2}}, a(t)^{\frac{\beta(t)}{2}}\right)\,. \tag{20}$$

  In this case, we obtain

$$ds^2_{\bar{\gamma}} = N\left(-4ada^2 + a^3 d\beta^2\right), \quad M(a,\beta) = \frac{a^3(t)}{N}. \tag{21}$$

In the present work we will focus on the static spherically-symmetric metric deriving new spherically symmetric solutions for $f(T)$ gravity. In particular, we look for Noether symmetries in order to reveal the existence of analytical solutions.

IV. THE NOETHER SYMMETRY APPROACH FOR $f(T)$ GRAVITY

The aim is now to extend results in [18] and [20] by applying the Noether Symmetry Approach [34] to a general class of $f(T)$ gravity models where the corresponding Lagrangian of the field equations is given by Eq.(8). First of all, we perform the analysis for arbitrary spacetimes, and then we focus on static spherically-symmetric geometries.

A. Searching for Noether point symmetries in general spacetimes

The Noether symmetry condition for the Lagrangian (8) is given by

$$X^{[1]}L + L\xi' = g', \tag{22}$$

where the generator $X^{[1]}$ is written as

$$X^{[1]} = \xi\left(\tau, x^k, T\right)\partial_\tau + \eta^k\left(\tau, x^k, T\right)\partial_i + \mu\left(\tau, x^k, T\right)\partial_T + \left(\eta'^i - \xi' x'^i\right)\partial_{x'^i}. \tag{23}$$

[2] Note that, in general, one can choose a non-diagonal vierbein, giving rise to the same metric through (1). However for the sake of simplicity, we remain in the diagonal case which is capable of revealing the main features of the solutions [13, 16, 18].

For each term of the Noether condition (22) for the Lagrangian (8) we obtain

$$\begin{aligned} X^{[1]}L \;=\; & 2f_T\bar{g}_{ij,k}\eta^k x'^i x'^j + M_{,k}\eta^k\left(f - Tf_T\right) \\ & +2f_{TT}\mu\bar{g}_{ij}x'^i x'^j - Mf_{TT}\mu \\ & +4f_T\bar{g}_{ij}x'^i\left(\eta^j_{,\tau} + \eta^j_{,k}x'^k + \eta^j_{,T}T'\right. \\ & \left.-\xi_{,\tau}x'^j - \xi_{,k}x'^j x'^k - \xi_{,T}x'^j T'\right), \end{aligned}$$

$$\begin{aligned} L\xi' = & \left[2f_T\bar{g}_{ij}x'^i x'^j + M\left(x^i\right)\left(f - Tf_T\right)\right] \\ & \cdot\left(\xi_{,\tau} + \xi_{,k}x'^k + \xi_{,T}T'\right), \end{aligned}$$

$$g' = g_{,\tau} + g_{,k}x'^k + g_{,T}T' \ .$$

Inserting these expressions into (22) we find the Noether symmetry conditions

$$\xi_{,k} = 0\ ,\ \xi_{,T} = 0\ ,\ g_{,T} = 0\ ,\ \eta_{,T} = 0, \tag{24}$$

$$4f_T\bar{\gamma}_{ij}\eta^k_{,\tau} = g_{,k}, \tag{25}$$

$$M_{,k}\eta^k\left(f - Tf_T\right) - MTf_{TT}\mu + \xi_{,\tau}M\left(f - Tf_T\right) - g_{,\tau} = 0, \tag{26}$$

$$2f_T\bar{\gamma}_{ij,k}\eta^k + 2f_{TT}\mu\bar{\gamma}_{ij} + 4f_T\bar{\gamma}_{ij}\eta^j_{,k} - 2f_T\bar{\gamma}_{ij}\xi_{,\tau} = 0\ . \tag{27}$$

Notice that conditions $\eta_{,T} = g_{,T} = 0$ imply, through Eq.(25), that $\eta^k_{,\tau} = g_{,k} = 0$. Also, Eq.(27) takes the form

$$L_\eta\bar{\gamma}_{ij} = \left(\xi_{,\tau} - \frac{f_{TT}}{f_T}\mu\right)\bar{\gamma}_{ij}, \tag{28}$$

where $L_\eta\bar{\gamma}_{ij}$ is the Lie derivative with respect to the vector field $\eta^i(x^k)$. Furthermore, from (28) we deduce that η^i is a Conformal Killing Vector of the metric $\bar{\gamma}_{ij}$, and the corresponding conformal factor is

$$2\bar{\psi}\left(x^k\right) = \xi_{,\tau} - \frac{f_{TT}}{f_T}\mu = \xi_{,\tau} - S(\tau, x^k)\ . \tag{29}$$

Finally, utilizing simultaneously Eqs.(26), (28), (29) and the condition $g_{,\tau} = 0$, we rewrite (26) as

$$M_{,k}\eta^k + \left[2\bar{\psi} + \left(1 - \frac{Tf_T}{f - Tf_T}\right)S\right]M = 0\ . \tag{30}$$

Considering that $S = S(x^k)$ and using the condition $g_{,\tau} = 0$, we acquire $\xi_{,\tau} = 2\bar{\psi}_0$, $\bar{\psi}_0 \in \mathbb{R}$ with $S = 2(\bar{\psi}_0 - \bar{\psi})$. At this point, we have to deal with the following two situations:

Case 1. In the case of $S = 0$, the symmetry conditions are

$$\begin{aligned} & L_\eta\bar{\gamma}_{ij} = 2\bar{\psi}_0\bar{\gamma}_{ij}, \\ & M_{,k}\eta^k + 2\bar{\psi}_0 M = 0, \end{aligned} \tag{31}$$

implying that the vector $\eta^i(x^k)$ is a Homothetic Vector of the metric $\bar{\gamma}_{ij}$. The latter means that for arbitrary $f(T) \neq T^n$ functional forms, our dynamical system could possibly admit extra (time independent) Noether symmetries.

Case 2. If $S \neq 0$ then Eq. (30) immediately leads to the following differential equation

$$\frac{Tf_T}{f - Tf_T} = C, \tag{32}$$

which has the solution

$$f(T) = T^n, \quad C \equiv \frac{n}{1-n}\ . \tag{33}$$

In this context, $\eta^i(x^k)$ is a Conformal Killing Vector of $\bar{\gamma}_{ij}$, and the symmetry conditions become

$$\begin{aligned} & L_\eta\bar{\gamma}_{ij} = 2\bar{\psi}\bar{\gamma}_{ij}, \\ & M_{,k}\eta^k + \left[2\bar{\psi} + (1 - C)\,S\right] = 0, \end{aligned} \tag{34}$$

with $S = 2(\bar{\psi}_0 - \bar{\psi})$.

Collecting the above results we can formulate the following proposition:

Lemma:

The general autonomous Lagrangian

$$L\left(x^k, x'^k, T\right) = 2f_T\bar{\gamma}_{ij}\left(x^k\right)x'^i x'^j + M\left(x^k\right)\left(f - Tf_T\right)$$

admits extra Noether symmetries as follows:

1. *If $f(T)$ is an arbitrary function of T, then the symmetry vector is written as*

   $$X^{[1]} = \left(2\psi_0\tau + c_1\right)\partial_\tau + \eta^i\left(x^k\right)\partial_i,$$

   where $\eta^i\left(x^k\right)$ is a Homothetic Vector of the metric $\bar{\gamma}_{ij}$ and the following condition holds

   $$M_{,k}\eta^k + 2\bar{\psi}_0 M = 0\ .$$

   Note that if η^i is a Killing Vector (or Homothetic Vector) then $\psi_0 = 0$ (or $\psi_0 = 1$).

2. *If $f(T)$ is a power law, namely T^n, then we have the extra symmetry vector*

   $$X^{[1]} = \left(2\bar{\psi}_0\tau\right)\partial_\tau + \eta^i\left(x^k\right)\partial_i + \frac{\left(2\bar{\psi}_0 - 2\bar{\psi}\right)}{C}T\partial_T,$$

   where $C = \frac{n}{1-n}$, η^i is a Conformal Killing Vector of the metric $\bar{\gamma}_{ij}$ with conformal factor $\bar{\psi}\left(x^k\right)$ and the following condition holds

   $$M_{,k}\eta^k + \left[2\bar{\psi} + (1 - C)\,S\right] = 0,$$

   with $S = 2(\bar{\psi}_0 - \bar{\psi})$.

   In both cases the corresponding gauge function is a constant.

B. Noether symmetries of the field equations in static spherically symmetric spacetimes

Let us now apply the results of the general Noether analysis of the previous subsection, to the specific case of static spherically-symmetric geometry, which is the subject of interest of the present work. Thus, from now on we focus on the metric (13), that is the vierbein (14).

TABLE I: Noether symmetries and integrals for arbitrary $f(T)$.

$N(a,b)$	Symmetry	Integral
$\frac{1}{a^3}N_1\left(a^2b\right)$	$-\frac{a}{2b^3}\partial_a+\frac{1}{b^2}\partial_b$	$\frac{N_1\left(a^2b\right)}{2a^3b^2}\left(2ba'+ab'\right)f_T$
$N_2\left(b\sqrt{a}\right)$	$-2a\partial_a+b\partial_b$	$N_2\left(b\sqrt{a}\right)\left(b^2a'-abb'\right)f_T$
$aN_3\left(b\right)$	$\frac{1}{ab}\partial_a$	$N_3\left(b\right)b'f_T$

Armed with the general expressions provided above, we can deduce the Noether algebra of the metric (15). In particular, the Lagrangian (8) and the Hamiltonian (9) become

$$L=2f_TN\left(2ba'b'+ab'^2\right)+M(a,b)\left(f-f_TT\right), \quad (35)$$

$$H=2f_TN\left(2ba'b'+ab'^2\right)-M(a,b)\left(f-f_TT\right)\equiv 0\ , \quad (36)$$

where $M(a,b)$ is given by (15). As one can immediately deduce, TEGR and thus General Relativity is restored as soon as $f(T)=T$, while if $N=1$, $\tau=r$ and $ab=1$ we fully recover the standard Schwarzschild solution.

Applying the results of the previous subsection in this specific case of static spherically-symmetric geometry, we determine all the functional forms of $f(T)$ for which the above dynamical system admits Noether point symmetries beyond the trivial one ∂_τ related to the energy, and we summarize the results in Tables I and II. Thus, we can use the obtained Noether integrals in order to classify the analytical solutions.

V. NEW CLASSES OF ANALYTICAL SOLUTIONS

Using the Noether symmetries and the corresponding integral of motions obtained in the previous section, we can extract all the static spherically-symmetric solutions of $f(T)$ gravity. We stress that, in this way, we obtain new solutions, that could not be obtained by the standard methods applied in [13–16].

Without loss of generality, we choose the conformal factor $N(a,b)$ such as $N(a,b)=ab^2$ [or equivalently[3] $M(a,b)=1$]. In order to simplify the current dynamical problem, we consider the coordinate transformation

$$b=(3y)^{\frac{1}{3}}\ ,\qquad a=\sqrt{\frac{2x}{(3y)^{\frac{1}{3}}}}\ . \quad (37)$$

Substituting the above variables into the field equations (11), (12), (36) we immediately obtain

$$x''+\frac{f_{TT}}{f_T}x'T'=0, \quad (38)$$

$$y''+\frac{f_{TT}}{f_T}y'T'=0, \quad (39)$$

$$H=4f_Tx'y'-\left(f-Tf_T\right), \quad (40)$$

while the torsion scalar (5) is given by

$$T=4x'y'\ . \quad (41)$$

Finally, the generalized Lagrangian (8) acquires the simple form

$$L=4f_Tx'y'+\left(f-Tf_T\right)\ . \quad (42)$$

Since the analysis of the previous subsection revealed two classes of Noether symmetries, namely for arbitrary $f(T)$, and $f(T)=T^n$, in the following subsections we investigate them separately. We would like to mention that the solutions provided below have been extracted under the assumption $f_{TT}\neq 0$, that is when $f(T)$ is not a linear function of T. Therefore, our solutions cannot be extrapolated back to the GR solutions where $f(T)=T$ (additionally note that these two cases exhibit different phase space and different Noether symmetries, and thus the obtained solutions do not always have a $f_{TT}\to 0$ limit).

A. Arbitrary $f(T)$

In the case where $f(T)$ is arbitrary, a "special solution" of the system (38)-(41) is

$$x\left(\tau\right)\ =\ c_1\tau+c_2, \quad (43)$$
$$y\left(\tau\right)\ =\ c_3\tau+c_4, \quad (44)$$

and the Hamiltonian constraint ($H=0$) reads

$$4c_1c_3\frac{df}{dT}|_{T=4c_1c_3}-f+T\frac{df}{dT}|_{T=4c_1c_3}=0, \quad (45)$$

where $T=4c_1c_3$, and $c_{1,..,4}$ are integration constants. We mention that the current solution is just one special solution in the case of arbitrary, non-linear $f(T)$, which is indeed characterized by a constant $T=4c_1c_3$. Definitely, the general solution will not have constant T.

Utilizing (37), (43) and (44), we get

$$b\left(\tau\right)=3^{\frac{1}{3}}\left(c_3\tau+c_4\right)^{\frac{1}{3}}\ ,$$
$$a\left(\tau\right)=\frac{\sqrt{6}}{3^{\frac{2}{3}}}\left(c_1\tau+c_4\right)^{\frac{1}{2}}\left(c_3\tau+c_4\right)^{\frac{1}{6}}\ . \quad (46)$$

[3] Since the space is empty, the field equations are conformal invariant, therefore the results are similar for an arbitrary function $N(a,b)$ [35].

TABLE II: Extra Noether symmetries and integrals for $f(T) = T^n$ with $C = \frac{n}{1-n}$. The last four lines correspond to the special case where $n = 1/2$. Notice, that $\bar{\psi}_{5-7}$ are the conformal factors defined as $\bar{\psi} = \frac{1}{\dim \bar{\gamma}_{ij}} \eta^k_{;k}$. We notify that the power law case also admits the Noether symmetries of Table I.

$N(a,b)$	Symmetry	Integral
arbitrary	$2\bar{\psi}_0 \tau \partial_\tau + \frac{2\psi_0(C-1)}{2C+1} a\partial_a + \frac{2\bar{\psi}_0 - 2\bar{\psi}_4}{C} T\partial_T$	$2\psi_0 n \frac{C-1}{1+2C} abN(a,b) T^{n-1} b'$
	$-2a\partial_a + b\partial_b - \frac{2\bar{\psi}_5}{C} T\partial_T$	$nN(a,b) T^{n-1} \left(b^2 a' - abb'\right)$
	$-\frac{a}{2} b^{-\frac{3(1+2C)}{4C}} \partial_a + b^{-\frac{3+2C}{4C}} \partial_b - \frac{2\bar{\psi}_6}{C} T\partial_T$	$\frac{n}{2} N(a,b) T^{n-1} \left(2b^{\frac{2C-3}{4C}} a' + ab^{-\frac{3+2C}{4C}} b'\right)$
	$a^{-\frac{1}{2C}} b^{-\frac{1+2C}{4C}} \partial_a - \frac{2\bar{\psi}_7}{C} T\partial_T$	$N(a,b) n a^{-\frac{1}{2C}} b^{-\frac{1+2C}{4C}} T^{n-1} b'$
arbitrary	$2\bar{\psi}_0 \tau \partial_\tau + \frac{3\bar{\psi}_0}{2} a \ln\left(a^2 b\right) \partial_a + \frac{2\bar{\psi}_0 - 2\bar{\psi}'_4}{C} T\partial_T$	$\frac{3}{2}\psi_0 N(a,b) T^{-\frac{1}{2}} ab \ln\left(a^2 b\right) b'$
	$b\partial_b - \frac{2\bar{\psi}_5}{C} T\partial_T$	$\frac{1}{2} N(a,b) T^{-\frac{1}{2}} \left(b^2 a' + abb'\right)$
	$-a \ln(ab) \partial_a + b \ln b \partial_b - \frac{2\bar{\psi}_6}{C} T\partial_T$	$\frac{1}{2} N(a,b) T^{-\frac{1}{2}} b \left(b \ln b\; a' - a \ln a\; b'\right)$
	$a\partial_a - \frac{2\bar{\psi}_7}{C} T\partial_T$	$\frac{1}{2} N(a,b) T^{-\frac{1}{2}} ab\; b'$

For convenience, we can change variables from $b(\tau)$ to r according to the transformation $b(\tau) = r$, where r denotes the radial variable. Inserting this into the above equations, we conclude that the spacetime (13) in the coordinates (t, r, θ, ϕ) can be written as

$$ds^2 = -A(r)\, dt^2 + \frac{1}{c_3^2} \frac{1}{A(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{47}$$

with

$$A(r) = \frac{2c_1}{3c_3} r^2 - \frac{2c_\mu}{c_3 r} = \lambda_A \left(1 - \frac{r_\star}{r}\right) R(r), \tag{48}$$

and

$$R(r) = \left(\frac{r}{r_\star}\right)^2 + \frac{r}{r_\star} + 1. \tag{49}$$

In these expressions, we have defined $c_\mu = c_1 c_4 - c_2 c_3$, $\lambda_A = \left(\frac{8c_1 c_\mu^2}{3c_3^3}\right)^{1/3}$, and $r_\star = (\frac{3c_\mu}{c_1})^{1/3} = (\frac{3c_3\lambda_A}{2c_1})^{1/2}$ is a characteristic radius with the restriction $c_\mu c_1 > 0$.

As we can observe, if we desire to obtain a Schwarzschild-de Sitter-like metric we need to select the constant c_3 such as $c_3 \equiv 1$. On the other hand, the function $R(r)$ can be viewed as a distortion factor which quantifies the deviation from the pure Schwarzschild solution. Thus, the $f(T)$ gravity on small spherical scales $(r \to r_\star^+)$ tends to create a Schwarzschild solution.

In order to explore the singularity and horizon features of the obtained solutions we additionally calculate the Kretschmann scalar [from the metric (47) we calculate the Levi-Civita connection, then the Riemann tensor R^{abcd}, and finally the Kretschmann scalar $\mathcal{K} \equiv R^{abcd} R_{abcd}$], obtaining

$$\begin{aligned}
\mathcal{K} &= \frac{1}{r^4} \left[c_3^4 A_{,rr}^2 r^4 \right. \\
&\quad \left. + 4c_3^4 A_{,r}^2 r^2 + 4 - 8c_3^2 A(r) + 4c_3^4 A(r)^2\right] \\
&= \frac{4}{3r^4} \left\{4c_3 r \left[c_3 \left(2c_1^2 r^3 - 6c_1 c_\mu r^2 - 3c_2 + 6c_\mu^2 r\right)\right.\right. \\
&\quad \left.\left. + c_1(3c_4 - r)\right] + 3\right\}.
\end{aligned}$$

Notice, that in this section we use the general definition $\Xi_{,r} = d\Xi/dr$ and $\Xi_{,rr} = d^2\Xi^2/dr^2$. As we can observe, $\mathcal{K}$ is singular at the origin, and thus the origin corresponds to a physical singularity, similar to the Schwarzschild solution. As usual it is hidden behind a horizon at $r = r_\star$, in which the Kretschmann scalar is finite.

Let us now examine the remaining physical features of the above solution. Within this framework, for the observers, $u^i u_i = -1$, it is easy to show that the Einstein's tensor becomes

$$G^i_j = \mathrm{diag}\left(2c_1c_3 - \frac{1}{r^2}, 2c_1c_3 - \frac{1}{r^2}, 2c_1c_3, 2c_1c_3\right).$$

Hence, one can treat the problem at hand using an "effective" fluid, which can be seen as a dynamical consequence of $f(T)$ gravity. Therefore, from the 1+3 decomposition we can define the corresponding effective energy density, pressure, heat flux and traceless stress-tensor, as measured by the observer u^i, as

$$\rho_T = \frac{1}{(u^i u_i)^2} G_{ij} u^i u^j = -2c_1c_3 + \frac{1}{r^2}, \tag{50}$$

$$p_T = \frac{1}{3} h^{ij} G_{ij} = 2c_1c_3 - \frac{1}{3r^2}, \tag{51}$$

$$q^i = h^{ij} G_{jk} u^k = 0, \tag{52}$$

$$\pi^\theta_\theta = \pi^\phi_\phi = -\frac{1}{2}\pi^r_r = \frac{1}{3r^3}, \tag{53}$$

where

$$\pi_{ij} = (h^r_i h^s_j - \frac{1}{3} h_{ij} h^{rs}) G_{rs}, \tag{54}$$

and h_{ij} is the projective tensor

$$h_{ij} = g^{ij} - \frac{1}{(u^i u_i)} u^i u^j. \tag{55}$$

We mention that in the case of GR, namely when $f(T) = T$, such an effective fluid does not exist. In this sense, $f(T)$ gravity resembles to $f(R)$ gravity, where the fact that $f(R) \neq R$ gives rise to a curvature effective fluid [1].

From the above results, we deduce that the obtained effective energy momentum tensor is written as

$$T_{ij} = \hat{T}_{ij} + \tilde{T}_{ij} \tag{56}$$

where

$$\hat{T}_{ij} = \hat{\rho} u_i u_j + \hat{p} h_{ij} \tag{57}$$

$$\tilde{T}_{ij} = \tilde{\rho} u_i u_j + \tilde{p} h_{ij} + \pi_{ij} \ . \tag{58}$$

Notice that we have made the corresponding splitting $\rho_T = \hat{\rho} + \tilde{\rho}$ and $p_T = \hat{p} + \tilde{p}$ with $\hat{p} = -\hat{\rho} = 2c_1c_3$ and $\tilde{p} = -\frac{1}{3}\tilde{\rho} = -\frac{1}{3r^2}$. Therefore, we conclude that the effective dark energy fluid, due to the $f(T)$ terms, consists of two parts. The first, namely $\hat{T}_{ij}$, plays the role of a cosmological constant, which is associated with the de Sitter-Schwarzschild metric[4] with $\Lambda = 2c_1c_3$. The second part, namely $\tilde{T}_{ij}$ corresponds to a fluid with equation-of-state parameter equal to $-1/3$, and is a pure effect of the $f(T)$ structure.

In order to apply the above considerations for specific $f(T)$ forms, we consider the following viable $f(T)$, motivated by cosmology [5]:

- Exponential $f(T)$ gravity [36]:

  $$f(T) = T + f_0 e^{-f_1 T},$$

  where f_0 and f_1 are the two model parameters which are connected via (45), that is

  $$f_0 = \frac{4c_1c_3}{8f_1c_1c_3 + 1} \exp\left(4f_1c_1c_3\right).$$

- A sum of two different power law $f(T)$ gravity:

  $$f(T) = T^m + f_0 T^n,$$

  where from (45) we have

  $$f_0 = \frac{1-2m}{2n-1} \left(4c_1c_3\right)^{m-n} \ .$$

  Note that in the case of $m = 1$ we recover the $f(T)$ model by Bengochea & Ferraro [7].

Let us make a comment here. Recently, it has been showed [17] that the above viable $f(T)$ models may be written as perturbations around the concordance ΛCDM model, which means that the corresponding Hubble function of these $f(T)$ models can be given in terms of the ΛCDM Hubble parameter. Interestingly enough, in the current paper, we find that the spherically symmetric solutions of the above $f(T)$ models smoothly includes the Schwarzschild solution [see Eq.(48)]. Generally, we are interested in viable $f(T)$ models, since these models can describe the matter and dark energy eras, being consistent with the observational data (including Solar System tests), and finally they have stable perturbations. Although these necessary analyses have not yet been performed for all the available $f(T)$ models, a failure of a particular model to pass one of these tests is sufficient to exclude it. In this respect, we plan to investigate in a forthcoming paper the performance of our spherical solutions against the Solar System tests, aiming to impose constraints on the free parameters.

B. The case $f(T) = T^n$

Based on considerations at cosmological scales, it has been found by Basilakos et al. [20] that the $f(T) = T^n$ gravity models suffer for two basic problems. The first is associated with the fact that the deceleration parameter is constant, that is it never changes sign, and therefore the universe always accelerates or always decelerates, depending on the value of n. Secondly, the growth rate of cosmic structures remains always equal to unity, implying that the recent growth data disfavor the $f(T) = T^n$ gravity. Despite the above caveats, for completeness in this subsection we provide the analytical solutions for the spherically symmetric geometry. In the $f(T) = T^n$ case, the field Eqs. (11), (12), (36) and the torsion scalar (41) give rise to the following dynamical system:

$$T = 4x'y', \tag{59}$$

$$4nT^{n-1}x'y' - (1-n)\,T^n = 0, \tag{60}$$

$$x'' + (n-1)\,x'T^{-1}T' = 0, \tag{61}$$

$$y'' + (n-1)\,y'T^{-1}T' = 0\ . \tag{62}$$

It is easy to show that combining Eq.(59) with the Hamiltonian (60), we can impose constraints on the value of n, namely $n = 1/2$. Under this condition, solving the system of Eqs. (61) and (62) we arrive at the solutions

$$x(\tau) = \frac{\sigma(\tau)^3}{3} + c_\sigma, \tag{63}$$

$$y(\tau) = \frac{\sigma(\tau)^3}{3}, \tag{64}$$

where c_σ is the integration constant. Now using (37) we derive a,b as

$$b(\tau) = \sigma(\tau), \tag{65}$$

$$a(\tau) = \sqrt{\frac{2\left[\sigma^3(\tau) + 3c_\sigma\right]}{3\sigma(\tau)}}. \tag{66}$$

Using the coordinate transformation $\sigma(\tau) = r$, which implies $\tau = F(r)$ [with $F(\sigma(\tau)) = \tau$], and using simultaneously (65), the spherical metric (13) can be written as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2\left(d\theta + \sin^2\theta d\phi^2\right)\ , \tag{67}$$

[4] The de Sitter-Schwarzschild solution is $A(r) = 1 - \frac{2\alpha}{r} - \frac{\Lambda}{3}r^2$ with $G^i_j = \mathrm{diag}\left(\Lambda, \Lambda, \Lambda, \Lambda\right)$, $p = -\rho = -\Lambda$, $q_i = 0$ and $\pi_{ij} = 0$.

[5] The $f(T)$ models of Refs.[7, 36] are consistent with the cosmological data.

where

$$A(r)=\frac{2}{3}r^2+\frac{2c_\sigma}{r}=\lambda_A(1-\frac{r_\star}{r})R(r), \quad (68)$$

with $\lambda_A=\left(\frac{8c_\sigma^2}{3}\right)^{1/3}$ and

$$B(r)=\frac{F_{,r}^2}{A(r)r^4}. \quad (69)$$

The functional form of the distortion parameter $R(r)$ is given by relation (49), in which the characteristic distance becomes $r_\star=(-3c_\sigma)^{1/3}=\left(\frac{3\lambda_A}{2}\right)^{1/2}$, implying $c_\sigma<0$.

Furthermore, considering the comoving observers $\left(u^iu_i\right)=-1$, we can write the Einstein tensor components as

$$\begin{aligned} G_t^t &= -\frac{r}{3F_{,r}^3}\left[4rF_{,rr}\left(r^3+3c_\sigma\right)-2F_{,r}\left(7r^3+12c_\sigma\right)\right. \\ &+ \left.\frac{3}{r^3}F_{,r}^3\right] \\ G_r^r &= \frac{2r^4}{F_{,r}^2}-\frac{1}{r^2} \\ G_\theta^\theta &= G_\phi^\phi=-\frac{1}{3}\frac{r}{F_{,r}^3}\left[rF_{,rr}\left(4r^3+3c_\sigma\right)-F_{,r}\left(14r^3+6c_\sigma\right)\right]. \end{aligned}$$

Similarly, based on the first equalities of (50)-(54), we provide the corresponding fluid components

$$\rho_T=\frac{4r^2F_{,rr}}{F_{,r}^3}\left(\frac{1}{3}r^3+c_\sigma\right)-\frac{2r}{F_{,r}^2}\left(\frac{7}{3}r^3-4c_\sigma\right)+\frac{1}{r^2} \quad (70)$$

$$p_T=-\frac{2}{3}\frac{r^2F_{,rr}}{F_{,r}^3}\left(\frac{4}{3}r^3+c_\sigma\right)+\frac{2}{3}\frac{r}{F_{,r}^2}\left(\frac{17}{3}r^3+2c_\sigma\right)-\frac{1}{3r^2} \quad (71)$$

$$\pi_{,r}^r=\frac{2}{3}\frac{r^3F_{,rr}}{F_{,r}^2}\left(\frac{4}{3}r^3+c_\sigma\right)-\frac{2}{3}\frac{1}{F_{,r}^2}\left(\frac{8}{3}r^3+2c_\sigma\right)-\frac{2}{3r^2}$$

$$\pi_\theta^\theta=\pi_\phi^\phi=-\frac{1}{2}\pi_r^r$$

$$q^i=0$$

Finally, if we desire to construct an effective fluid that obeys a barotropic equation of state $p_T=(\gamma-1)\rho_T$ (frequently used in cosmological studies), then using Eqs.(70), and (71), we need to write $F_{,r}$ as

$$F_{,r}=\frac{1}{\sqrt{Z\left(r\right)}}J\left(r\right)^{\frac{3\gamma}{3\gamma-1}}r^2, \quad (72)$$

where

$$Z(r)=3\left(3\gamma-2\right)\int J\left(r\right)^{\frac{1}{3\gamma-1}}+F_1\ , \quad (73)$$

and

$$J(r)=(6\gamma-2)r^3+(18\gamma-15)c_\sigma. \quad (74)$$

From the above functions, it is clear that, in order to have a real solution, the corresponding γ parameter has to obey the restriction $\gamma>2/3$. To this end, inserting Eq.(72) into Eq.(69) we obtain

$$B(r)=\frac{J\left(r\right)^{\frac{6\gamma}{3\gamma-1}}}{Z\left(r\right)A(r)}, \quad (75)$$

where the function $A(r)$ is given by (68) and F_1 is the constant of integration.

We mention here that such further terms could describe interesting effects if coupled with an equation of state of the form $p_m=K_0\rho_m^\gamma$ where p_m and ρ_m are the relative quantities related to standard matter fluids. As in the case of $f(R)$ gravity, anomalous stars could be addressed by constructing modified Lané - Emden equations, where further geometric terms play a relevant role (see e.g. [38]). In particular, the above discussion could be useful in order to deal with anisotropic deformations of neutron star instead of searching for exotic form of matter [37]. As discussed in [39, 40] for $f(R)$ gravity, geometric pressure terms, inserted in the Tolman-Oppenheimer-Volkoff solution, could account for the larger effective masses of some neutron stars, recently observed [41, 42], that escape the standard GR interpretation.

VI. CONCLUSIONS

In Basilakos et al. [20] we have utilized the Noether symmetry method in order to investigate the main properties of the $f(T)$ modified gravity in the flat FLRW cosmology. In this work, we present a complete Noether symmetry analysis in the framework of $f(T)$ gravity. Specifically, considering $f(T)$ gravity embedded in the static spherically symmetric spacetime, we provide a full set of Noether symmetries for the related minisuperspaces. Then we compute new analytical solutions for various $f(T)$ models. Interestingly, we find that the $f(T)$ static spherically symmetric spacetime is written in terms of the well known Schwarzschild spacetime, modified by a distortion function that depends on a characteristic radius. We mention that the obtained solution classes are more general and cannot be obtained by the usual solutions methods. Obviously, the combination of the work by Basilakos et al. [20] with the current article provide a complete investigation of the Noether symmetry approach in $f(T)$ gravity at FLRW and spherical levels respectively.

From a genuine physical point of view, this means that $f(T)$ gravity could be a reliable approach in order to deal with several open issues in astrophysics and cosmology. In a forthcoming paper, we will consider in details such applications.

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